Graphons
A tool for the analysis of dynamical systems on large networks

Paolo Frasca

CNRS, GIPSA-lab, Grenoble

Scuola Superiore Meridionale
November 25, 2021
Outline

1. Large networks

2. What are graphons?
   - Statistical interpretation

3. Application to SIS epidemics
   - Networked SIS model
   - Stability
   - Sensitivity to noise

4. Conclusion

5. Sequel: Laplacians of graphs and graphons
Large networks
The challenges of Large Networks

Complex networks are often very large.

- The computational cost of algorithms for analysis and control can be prohibitive.
- Their structure is known with much uncertainty, because of
  - noise, partial observations and errors in data;
  - links (and nodes) changing with time

There is need for methods that “scale” well and cope with uncertainty.
What are graphons?
A graphon $W$ is a bounded symmetric function $W : [0, 1]^2 \to [0, 1]$.
A graphon can be interpreted as a continuous version of the adjacency matrix of a graph.
A *graphon* is the limit of a convergent sequence of graphs.
Statistical interpretation: Sampling from a graphon

A graphon can be used to generate random graphs of desired size by using a sampling method.

Step 1: Complete weighted graph $H$

$$[A_H]_{ij} := W(u_i, u_j), \ i, j \in \{1, \ldots, N\}$$

- Deterministic sampling: $\{u_i = \frac{i}{N}\}_{i=1}^{N}$
- Stochastic sampling, from a uniform distribution in $[0, 1]$: $\{u_i = U(i)\}_{i=1}^{N}$, $U(i)$ is the $i$-th order statistic of
Step 2: Simple graph $G$
Connect each pair of distinct vertices $i \neq j$ with probability $[A_H]_{ij}$, independently of the other edges.
Examples and classes of graphons

- **Constant** graphons $W(x, y) = p$.
  This generates *Erdős-Rényi* graphs with connection probability $p$

- **Piecewise constant** graphons
  This is the *stochastic block model* for networks with communities

- **Piecewise Lipschitz** graphons

**Definition (L, K-Piecewise Lipschitz Graphon)**

There exists a constant $L$ and a sequence of non-overlapping intervals $I_k = [\alpha_{k-1}, \alpha_k)$ defined by $0 = \alpha_0 < \cdots < \alpha_{K+1} = 1$, for a (finite) $K \in \mathbb{N}$ such that for any $k, l$, any set $I_{kl} = I_k \times I_l$ and pairs $(x, y) \in I_{kl}$, $(x', y') \in I_{kl}$, we have:

$$|W(x, y) - W(x', y')| \leq L \left( |x - x'| + |y - y'| \right).$$

$L$ is Lipschitz constant, $K + 1$ is the number of “pieces”
SIS epidemic model
Modeling the spreading of an infectious disease among a population of individuals: many different approaches, depending on the type of models and number of compartments (M, S, I, E, R, etc.).

**SIS model:** good for diseases where the infected agents can get infected again after recovering.

- **Susceptible (S):** healthy but susceptible to becoming infected.
- **Infected (I):** infected but able to recover.

\[
\dot{x} = -\delta x + \beta x(1 - x)
\]

where \(x \in [0, 1]\) is **infected fraction**, \(\delta\) is recovery rate, and \(\beta\) is infection rate.
Networked SIS dynamics

What about inhomogenous populations?

We can assume to have a network of (well-mixed) sub-populations. The dynamics of each node $i \in \{1, \ldots, N\}$ is modeled as:

$$\dot{x}_i(t) = -\delta x_i(t) + \sum_{j=1}^{N} a_{ij} \beta x_j(t)(1 - x_i(t)), \quad x_i(t) \in [0, 1],$$

The model of the whole network can be expressed as:

$$\dot{x}(t) = (\beta A - \delta I)x(t) - \beta X(t)Ax(t), \quad X(t) = \text{diag}(x_1(t), \ldots, x_N(t)).$$
Stability conditions

The disease-free equilibrium $x = 0$ is globally asymptotically stable if

$$R_0 := \lambda_1(A) \frac{\beta}{\delta} < 1$$

where $\lambda_1(A)$ is the largest eigenvalue (and spectral radius) of $A$. 
The disease-free equilibrium \( x = 0 \) is globally asymptotically stable if

\[
R_0 := \frac{\lambda_1(A) \beta}{\delta} < 1
\]

where \( \lambda_1(A) \) is the largest eigenvalue (and spectral radius) of \( A \).

For a sequence of graphs parametrized by their size \( N \), we would obtain

\[
\delta_N > \beta_N \lambda_1(A_N).
\]

We want to replace this condition with a condition involving the spectral radius of the graphon!
Spectra of graphons

Each graphon defines an integral operator $T_W : L^2[0,1] \to L^2[0,1]$

$$(T_W f)(x) = \int_0^1 W(x,y)f(y)\,dy$$

This operator is Hilbert-Schmidt, continuous and compact, with discrete spectrum where 0 is the only accumulation point.

The spectral radius is $\|T_W\| = \sup_{f \in L^2[0,1], \|f\|=1} \|T_W f\|$

Furthermore, we have convergence of normalized spectra: let $\lambda_1 \geq \lambda_2 \geq \cdots$ and, for a graphon of rank $M$,

$$\lambda_i(T_W) = \begin{cases} \text{non-zero eigenvalues} & i = 1, \ldots, M \\ 0 & i = M + 1, \ldots, N. \end{cases}$$

Then,

$$\frac{1}{N}\lambda_i(A_N) \to \lambda_i(T_W)$$
Theorem (Stability condition)

Let $W$ be a piecewise Lipschitz graphon and $G$ a graph with $N$ nodes sampled from $W$. Then, for $N$ large enough, the disease-free state is asymptotically stable with probability $1 - \nu$ if:

\[ \delta_N > N\beta_N \left( \|T_W\| + \phi(N) \right), \]

where $\nu \in (Ne^{-N/5}, e^{-1})$ and $\phi(N) := \sqrt{\frac{4\log(2N/\nu)}{N}} + \frac{2\sqrt{L^2 - K^2 + KN}}{N}$

Note: So long as $\nu = \Omega(1/N^a)$, $\phi(N) = O\left(\sqrt{\frac{\log N}{N}}\right)$

The stability condition becomes

\[ \frac{N\beta_N}{\delta_N} \|T_W\| < 1 \]
We consider the presence of additive noise in the linearized dynamics, which can represent mobility of the population or other phenomena that are not included in the original model:

\[ \dot{x}(t) = (\beta A - \delta I)x(t) + n(t), \]

where \( n(t) \in \mathbb{R}^N \) is a stochastic noise process.

A measure of sensitivity is the asymptotic mean-square error:

\[ J_{\text{noise}} := \lim_{t \to \infty} \frac{1}{N} \mathbb{E} \left[ \|x(t)\|_2^2 \right], \]

which can be used to quantify how much the epidemics react to noise in the neighborhood of the equilibrium.
Proposition

If the stability condition is satisfied and the noise process has zero mean and autocorrelation function \( \mathbb{E}[n(t)n(t - \xi)^T] = \sigma^2 \delta(\xi)I \), then, the noise index can be expressed as:

\[
J_{\text{noise}} = \frac{\sigma^2}{2N} \sum_{i=1}^{N} \frac{1}{\delta - \beta \lambda_i(A)}.
\]

We mimic this formula using the spectrum of the graphon

\[
J_{\text{noise}, W,N} := \frac{\sigma^2}{2N} \sum_{i=1}^{N} \frac{1}{\delta_N - \beta_N N \lambda_i(T_W)}
\]
Asymptotic equivalence

**Theorem (Approximation bound)**

Let $W$ be a piecewise Lipschitz graphon with finite rank $M$ and $G$ a simple graph with $N \geq M$ nodes sampled from $W$. If $\delta_N$ and $\beta_N$ satisfy

$$
\frac{\beta_N}{\delta_N} = \frac{1}{N} \frac{\bar{\beta}}{\bar{\delta}} \quad \text{and} \quad \delta_N > N \beta_N (\|T_W\| + \phi(N)),
$$

then for $N$ large enough, with probability at least $1 - \nu$:

$$
\left| J^{\text{noise}}_G - J^{\text{noise}}_{W,N} \right| \leq \frac{1}{N^{3/4}} \frac{\sigma^2 \bar{\beta}}{4/2(\bar{\delta} - \bar{\beta})} \sqrt{N \log(2N/\nu) + \sqrt{L^2 - K^2} + KN},
$$

where $\nu \in (Ne^{-N/5}, e^{-1})$ and $\phi(N) := \sqrt{\frac{4\log(2N/\nu)}{N}} + \frac{2\sqrt{L^2 - K^2 + KN}}{N}$.

$$
\left| J^{\text{noise}}_G - J^{\text{noise}}_{W,N} \right| = \mathcal{O}\left(\frac{(\log N)^{1/4}}{N^{1/2}}\right) \quad \text{with probability 1}
$$
Example: Stochastic Block Model

\[
\Delta \text{noise} = J_{G}^{\text{noise}} - J_{W,N}^{\text{noise}}
\]

Relative error \[
\frac{\Delta \text{noise}}{J_{G}^{\text{noise}}} = O\left(\frac{(\log N)^{1/4}}{N^{1/2}}\right)
\]

\[
W_{SB} = \begin{bmatrix}
0.9 & 0.7 & 0.6 & 0.5 & 0.2 \\
0.7 & 0.4 & 0.1 & 0.3 & 0.1 \\
0.6 & 0.1 & 0.5 & 0.9 & 0.8 \\
0.5 & 0.3 & 0.9 & 0.5 & 0.5 \\
0.2 & 0.1 & 0.8 & 0.5 & 0.7 \\
\end{bmatrix}
\]

blue: computed – red: theoretical bound
Graphons are an excellent tool to derive approximate conditions and to approximately compute quantities that depend on large networks.

Examples: stability conditions & sensitivity index of SIS epidemics.

Beyond these examples, many interesting things can be done:

- Work on Laplacian matrix and Laplacian spectrum
- Study dynamics on the graphons (e.g. SIS)
- Compare dynamics on graphs vs dynamics on graphons
- Use graphons for control design
- Extend these results to sparse networks, i.e. with degree \( o(N) \)
Laplacians
The main difficulty in dealing with Laplacians is that the Laplacian operator associated to a graphon is not compact. Nevertheless, some results can be found, for instance about the Kirchoff index (a.k.a. average effective resistance), which is a measure of network connectivity:

$$R^\text{ave}_N := \frac{1}{N} \sum_{i=2}^{N} \frac{1}{\lambda_i(L_N)}$$

with Laplacian matrix $L_N = D_N - A_N$, degree matrix $D_N$. 
The main difficulty in dealing with Laplacians is that the Laplacian operator associated to a graphon is not compact. Nevertheless, some results can be found, for instance about the Kirchoff index (a.k.a. average effective resistance), which is a measure of network connectivity:

\[ R^\text{ave}_{N} := \frac{1}{N} \sum_{i=2}^{N} \frac{1}{\lambda_i(L_N)} \]

with Laplacian matrix \( L_N = D_N - A_N \), degree matrix \( D_N \).

We mimic this formula by

\[ R^\text{ave}_{W,N} := \frac{1}{N} \int_{0}^{1} \frac{dx}{d(x)} , \quad \text{with} \quad d(x) := \int_{0}^{1} W(x,y) \, dy \quad \text{(degree)} \]

because the degree matrix dominates the adjacency matrix in dense graphs.
Approximation result

Theorem

Let $R_{N}^{\text{ave}}$ be the average effective resistance of a graph $G_N$ sampled from a piecewise Lipschitz graphon $W$ with minimum $\eta_W := \inf W(x, y) > 0$. Then, for $N$ large enough satisfying $\frac{\log(2N/\nu)}{N} < \frac{\eta_W^2}{1 + 2\eta_W}$, with probability at least $1 - 3\nu$, it holds true that:

$$\left| R_{N}^{\text{ave}} - R_{W,N}^{\text{ave}} \right| \leq \frac{1}{N(\eta_W - \gamma(N))} \left( \frac{1}{N} + \frac{\phi(N)}{\delta_W} + \frac{4\sqrt{2} \sqrt{\|TW\| + \phi(N)}}{N^{1/4} (\eta_W - \varphi(N))} \right),$$

where $\phi(N) := \sqrt{\frac{4\log(2N/\nu)}{N}} + 2\sqrt{(L^2 - K^2)b_N^2 + K b_N}$, $\delta_W = \inf d(x)$

$b_N := \frac{1}{N} + \sqrt{\frac{8\log(N/\nu)}{N+1}}$, $\gamma(N) = \sqrt{\frac{\log(2N/\nu)}{N\eta_W}}$ and $\varphi(N) := \left( \frac{1}{\sqrt{\eta_W}} + 2 \right) \sqrt{\frac{\log(2N/\nu)}{N}}$

$$\left| \frac{R_{N}^{\text{ave}} - R_{W,N}^{\text{ave}}}{R_{N}^{\text{ave}}} \right| = O \left( \frac{\log N}{N} \right)^{1/4}$$
Lipschitz graphon $W(x, y) = 1 - 0.8xy$: 

$$\frac{|R_N^{\text{ave}} - R_{W,N}^{\text{ave}}|}{R_N^{\text{ave}}} = O\left(\frac{1}{N^{1/4}}\right)$$
Some references

Our work

SIS on graphons

Useful references on graphons
Final acknowledgements

Renato Vizuete

Federica Garin

Funded by French National Science foundation: ANR-11-LABX-0025-01, ANR-18-CE40-0010