Evolution by curvature of networks in the plane

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Curve Shortening Flow

YouTube – https://www.youtube.com/watch?v=wHfpacPLHIA

Interactive – https://a.carapetis.com/csf
Networks of Curves
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... flowing
Joint project with
- Matteo Novaga & Vincenzo Tortorelli, 2003 – 2005
- Annibale Magni & Matteo Novaga, 2010 – 2014
- Matteo Novaga, Alessandra Pluda & Felix Schulze, 2014 –
- Pietro Baldi & Emanuele Haus, 2015 –

After the works of Huisken et al. about the mean curvature flow of curves and hypersurfaces, weak definitions of mean curvature flow of any merely closed set in the Euclidean space appeared.

We were interested in the study of the possibly “least singular” set: a network of curves in the plane. This is clearly a (toy) model for the time evolution of the interfaces of a multiphase planar system where the energy is given only by the total length of such interfaces.

Even if it is still possible to continue to use several of the ideas and techniques of the “parametric” (smooth, classical) approach (differential geometry/PDEs), some extra variational “weak” methods are needed, due to the presence of the multi–points.
We say that a network \textit{moves by curvature} if any of its time–dependent curves $\gamma^i : [0, 1] \times (0, T) \to \mathbb{R}^2$ satisfy

$$\gamma^i_t(x, t)^{\perp} = k^i(x, t) = \frac{\langle \gamma^{i xx}(x, t) | \nu^i(x, t) \rangle}{|\gamma^i_x(x, t)|^2} \nu^i(x, t) = \left( \frac{\gamma^{i xx}(x, t)}{|\gamma^i_x(x, t)|^2} \right)^\perp$$

for every $x \in [0, 1]$ and $t \in (0, T)$.

The \textit{normal component} of the velocity at every point is given by the curvature vector of the curve (till the endpoints of the curves).

With the right choice of the tangential component of the velocity the problem becomes a non–degenerate system (with several geometric properties) of \textit{quasilinear parabolic partial differential equations}.

This evolution can be seen as the \textit{geometric gradient flow} of the \textit{length functional}, that is, the sum of the lengths of all the curves of the network.
Some easy observations from the simulations

The area of the regions bounded by more than 6 edges grows, less than 6 edges decreases.

With the exception of the times when a structural change happens (vanishing of a curve or of a region), there are only triple junctions and the three concurring curves form angles of 120 degrees. We call such a network regular.

If no region is collapsing, the geometric changes are only given by pairs of triple junctions colliding (the curve connecting them vanishes - its length goes to zero), producing a 4–point in the network.

Immediately after such a collision of two triple junctions, the network becomes again regular (only triple junctions, with curves forming angles of 120 degrees): a new pair of triple junctions "emerges" from every 4–point.

Actually, despite the (apparently) simple problem/behavior/statements, to show in a mathematically satisfactory way these observations, a lot of "technology" from analysis and geometry is needed.
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Actually, despite the (apparently) simple problem/behavior/statements, to show in a mathematically satisfactory way these observations, a lot of “technology” from analysis and geometry is needed.
We started dealing with the local problem, that is, the study of the evolution by curvature of the simplest network of three non–intersecting curves with fixed endpoints and a single triple junction with angles of 120 degrees, called a regular triod.

For any initial regular smooth triod there exists a smooth flow by curvature in a positive maximal time interval. Moreover, the evolving triod stays regular.

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It can be seen as a “local regularity result” for the flow of a general regular network.
A regular network is given by a finite family of non-intersecting curves such that there are only a finite number of triple junctions with angles of 120 degrees between the concurring curves.

\[
\begin{align*}
O_1 & \quad O_2 & \quad O_3 & \quad O_4 & \quad O_5 & \quad O_6 & \quad O_7 & \quad O_8 \\
P_1 & \quad P_2 & \quad P_3 & \quad P_4 & \quad P_5 & \quad P_6 & \quad P_7 & \quad P_8
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At the maximal (singular) time \(T\) at least one (or both) of the following two conditions holds:

- The curvature is unbounded as \(t \to T\).
- The length of one (or more) of the curves of the network goes to zero (change of structure/topology).
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Even if there is no collapse of curves (or regions), so the topological structure of the network is not going to change, the connection between the “local regularity” (the special case of a triod) to the “global regularity” (general network) is not direct. The main tool is blow–up analysis (after integral estimates) and, in order to get regularity of the flow, one has to exclude that curves with multiplicity larger than one appear in the limit of rescaled networks (which are *shrinkers* – networks self–similarly moving by curvature).
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\textit{Every possible limit of rescaled networks is a network with multiplicity one.}
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Theorem

Assuming M1, if none of the lengths of the curves of an evolving regular network goes to zero as \( t \to T \), then \( T \) cannot be a singular time.

Hence, to proceed in the analysis, we have to deal with the situation when the length of at least one curve of the network goes to zero as \( t \to T \).

There are two cases:

- The curvature stays bounded.
- The curvature is unbounded as \( t \to T \).
The analysis in the first case (bounded curvature $\implies$ no collapse of regions) consists in understanding the possible limit networks that can arise as $t \to T$ and finding out how to continue the flow (if possible).

It can be shown that, as $t \to T$, such limit network, is unique. Anyway, it can be non-regular since multiple points can appear.

If the collapsing curve is not one of the family containing the fixed boundary points (boundary curves), we have the following result.

Lemma

If $M_1$ is true, every interior vertex of such limit network either is a regular triple junction or it is a 4-point where the four concurring curves have opposite unit tangents in pairs and form angles of $120/60$ degrees among them.

If the collapsing curve is one of the boundary curves, the flow stops.

Otherwise, is it possible to "restart" the flow? Being this limit network non-regular since it has also 4-junctions, the previous short time existence theorem does not apply.
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Being this limit network non–regular since it has also 4–junctions, the previous short time existence theorem does not apply.
Theorem (T. Ilmanen, A. Neves, F. Schulze – 2014)

For any initial network of non-intersecting curves there exists a (possibly non-unique) Brakke flow by curvature in a positive maximal time interval such that for every positive time the evolving network is smooth and regular.
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So, possibly losing the uniqueness of the flow (necessary – think of a cross), we are able to start the flow also for an initial non-regular network. Moreover, if the multiplicity–one conjecture is true, we know how to continue the flow till the curvature of the curves of the network stays bounded.
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The local description of a “standard” transition.
A “standard” transition for a Θ–shaped network (double cell).
The second situation, when the curvature is unbounded and some curves are vanishing, can be faced again with blow-up methods, but in general, even if $\textbf{M1}$ is true, there can be several possible limits of rescaled networks, making the classification quite difficult. Then, the (local) structure (topology) of the evolving network plays an important role in the analysis.
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**Theorem**

*If M1 holds and the network is a tree (no loops), the curvature is uniformly bounded during the flow, hence the only “singularities” (after which we can restart the flow as we described before) are given by the collapse of a curve with only two triple junctions going to coincide.*
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WORK IN PROGRESS – Conjecture

If $M_1$ holds and the network is general, as $t \to T$, there exists a unique limit non-regular network, with multiple points or even with triple junctions not satisfying 120 degrees condition, to which the previous “restarting theorem” can be applied to continue the flow.
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The number of singular times is finite. If no boundary curves collapse, the flow is definitely smooth and the evolving network converges (asymptotically) to a Steiner configuration connecting the fixed endpoints.
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**Theorem (CM, M. Novaga, A. Pluda – 2015)**

- *If during the flow the triple–junctions stay uniformly far each other, then M1 is true.*
- *If the initial network has at most two triple junctions, then M1 is true.*
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- If during the flow the triple–junctions stay uniformly far each other, then M1 is true.
- If the initial network has at most two triple junctions, then M1 is true.

Then, as the classification of self–shrinking networks with at most two triple junctions is complete, some special cases of flows of networks with “few” triple junctions can be fully analyzed.
Only 1 triple junction
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The Triod – A. Magni, CM, M. Novaga, V. Tortorelli
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The Triod – A. Magni, CM, M. Novaga, V. Tortorelli

The Spoon – A. Pluda
The Eyeglasses
The Eyeglasses and... the Broken Eyeglasses
2 triple junctions – CM, M. Novaga, A. Pluda

The “Steiner”, Theta, Lens and Island

Diagram showing various configurations of triple junctions with labeled angles and points.
Open problems and research directions
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- Short time existence/uniqueness for special initial interfaces by Depner–Garcke–Kohsaka
- Short time existence/estimates for special initial interfaces by Schulze–White
Thanks for your attention