Global and cluster synchronization in complex networks and beyond

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What is synchronization?

https://www.youtube.com/watch?v=Aaxw4zbULMs
Metronomes do not synchronize

Metronomes do synchronize
The model of metronomes

• A single metronome on a mobile base

\[ \frac{d^2 \vartheta}{dt^2} + \frac{mr_{c,m} g}{I} \sin \vartheta + \varepsilon \left[ \left( \frac{\vartheta}{\vartheta_0} \right)^2 - 1 \right] \frac{d \vartheta}{dt} + \left( \frac{mr_{c,m} \cos \vartheta}{I} \right) \frac{d^2 x}{dt^2} = 0 \]

• Neglecting the damping of the base motion, the coupling term (for example for a two metronome system) can be found

\[ x = -\frac{m}{M + 2m} r_{c,m} (\sin \vartheta_1 + \sin \vartheta_2) \]

• It can be analytically proven that metronomes synchronize [J. Pantaleone, Am. J. Phys., 2002]
Systems that synchronize: a classical example

- Each firefly emits light flashes with a regular internal cycle.
- Each firefly adjusts its lighting frequency as a function of its neighbors.
Other examples

https://www.youtube.com/watch?v=eAXVa_XWZ8
https://www.youtube.com/watch?v=W-nTOo95Yy8
Model of coupled oscillators

- Equations

\[
\dot{x}_i = f(x_i) + \sigma \sum_{j=1}^{N} a_{ij} H(x_j - x_i) \quad i = 1, \ldots, N
\]

\[a_{ij} = 1 \text{ if the nodes are connected}\]
\[a_{ij} = 0 \text{ otherwise}\]

Adjacency matrix

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]
Model of coupled oscillators

• Equations

\[ \dot{x}_i = f(x_i) + \sigma \sum_{j=1}^{N} a_{ij} H(x_j - x_i) \quad i = 1, \ldots, N \]

- \( f \) is the dynamics of each uncoupled unit (units are identical), order \( n \)
- \( a_{ij} \) are the elements of the adjacency matrix
- \( H \) is a constant matrix (the inner coupling)
- \( \sigma \) is the coupling coefficient

• Equivalent formulation

\[ \dot{x}_i = f(x_i) - \sigma \sum_{j=1}^{N} L_{ij} H x_i \quad i = 1, \ldots, N \]
Model of coupled oscillators

• Equivalent model

\[ \dot{x}_i = f(x_i) - \sigma \sum_{j=1}^{N} L_{ij} H x_i \quad i = 1, \ldots, N \]

- \[ L_{ij} = -1 \] if the nodes are connected
- \[ L_{ij} = 0 \] otherwise
- \[ L_{ii} = d_i \]

Laplacian matrix (L=D-A)
Global synchronization

• All units asymptotically converge to the same trajectory \( x_1 = x_2 = \ldots = x_N = x_s \)

\[
\lim_{t \to \infty} \| x_i(t) - x_j(t) \| = 0, \quad \forall i, j
\]

• With diffusive coupling, the solution \( x_1 = x_2 = \ldots = x_N = x_s \) always exists!

\[
\begin{align*}
L \mathbf{1} &= 0 \\
\dot{x}_s &= f(x_s)
\end{align*}
\]

• Is this solution stable?
Stability of the synchronous state

Function characteristics of the unit and the way it is coupled (i.e. f and h)

- **Type I**: networks never synchronizable
- **Type II**: \( \sigma > \frac{\alpha_1}{\lambda_2} \)
- **Type III**: \( \sigma > \frac{\alpha_1}{\lambda_2} \) and Requires that: \( \frac{\lambda_N}{\lambda_2} < \frac{\alpha_2}{\alpha_1} \)

Synchronization

\[ \lambda_{\text{max}} (\sigma \lambda_h) < 0 \quad \forall \lambda_h \text{ eigenvalue of } L \]
From global to cluster synchronization

Global synchronization

Cluster synchronization
Symmetries in complex networks

Symmetries of an object: those operations that applied to it leave it unchanged
Symmetries of a graph form a group, can be represented as a permutation matrix $R_{gi}$

Adjacency and Laplacian matrix are invariant to graph symmetries

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_{g1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$R_{g2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$R_g A = AR_g \quad R_g L = LR_g$$
From symmetries to cluster synchronization

- Symmetries determine a partition of the network nodes into \( M \) groups (rigorously \textit{orbits}): \( V_1, V_2, ..., V_M \)
- Cluster synchronization

\[
\lim_{t \to \infty} \| x_i(t) - x_j(t) \| = 0, \quad \forall i, j \in V_h, \ h = 1, ..., M
\]

- Symmetry implies invariance of motion, existence of a non-trivial synchronization manifold
- Stability analysis: extended MSF-based approach
A control problem in cluster synchronization

- **Problem 1:** given a network with adjacency matrix $A$ and a target set of symmetries, finding a perturbation $\Delta A$ such that the new network, with adjacency matrix $A + \Delta A$ admits the given set of symmetries.

- **Problem 2:** given a network of identical dynamical units with connectivity $A + \Delta A$, stabilizing the desired CS pattern through a proper setting of the coupling strength $\sigma$

\[
\dot{x}_i = f(x_i) + \sigma \sum_{j=1}^{N} (a_{ij} + \Delta a_{ij}) H(x_j - x_i) \quad i = 1, \ldots, N
\]

A Sylvester equation for inducing symmetries in a network

• Let

\[ \tilde{A} = A + \Delta A \]

and

\[ R_1, \ldots, R_g \]

the set of target symmetries. Then, we have to find such that the following equations hold true:

\[ R_i (A + \Delta A) = (A + \Delta A) R_i, \quad i=1,\ldots,g \]
Equivalent optimization problem

\[ R_i (A + \Delta A) = (A + \Delta A) R_i, \quad i=1,\ldots,g \]

\[ \mathcal{R}_i \cdot \text{vec}(\Delta A) = \text{vec}(-R_iA + AR_i) \]

\[ \mathcal{R}_i = I_N \otimes R_i - R_i^T \otimes I_N \]

\[ \mathcal{R} \cdot \text{vec}(\Delta A) = B \]

Solutions

1. Moore-Penrose inverse

\[ \min \| \text{vec}(\Delta A) \|_2, \text{ subject to } \mathcal{R} \cdot \text{vec}(\Delta A) = B \]

\[ \text{vec}(\Delta A) = \mathcal{R}^\dagger B. \]
Equivalent optimization problem

\[ R_i (A + \Delta A) = (A + \Delta A) R_i, \quad i=1,...,g \]

\[ \mathcal{R}_i \cdot \text{vec}(\Delta A) = \text{vec}(-R_i A + AR_i) \]

\[ \mathcal{R}_i = I_N \otimes R_i - R_i^T \otimes I_N \]

\[ \mathcal{R} \cdot \text{vec}(\Delta A) = B \]

Solutions

2. Lasso (least absolute shrinkage and selection operator)

\[ \min \{ ||\mathcal{R} \cdot \text{vec}(\Delta A) - B||_2^2 + \beta ||\text{vec}(\Delta A)||_1 \} \]
Equivalent optimization problem

\[ R_i (A + \Delta A) = (A + \Delta A) R_i, \quad i=1,\ldots,g \]

\[ \mathcal{R}_i \cdot \text{vec}(\Delta A) = \text{vec}(-R_iA + AR_i) \]

\[ \mathcal{R}_i = I_N \otimes R_i - R_i^T \otimes I_N \]

\[ \mathcal{R} \cdot \text{vec}(\Delta A) = B \]

Solutions

3. Connectedness Preserving Optimization (CPO)

\[ \min \| \text{vec}(\Delta A) \|_2, \]

subject to \( \mathcal{R} \cdot \text{vec}(\Delta A) = B \) and to \( \Delta A_{ij} \geq 0, \forall i \neq j \)
Stability of the CS state

- Linearization + decoupling: transverse modes

\[
\delta y_\perp = \left[ \sum_{l=1}^{M} \tilde{E}^l \otimes Df(s_l) - \sigma \tilde{L}_\perp \sum_{l=1}^{M} \tilde{E}^l \otimes Dh(s_l) \right] \delta y_\perp,
\]

- CS is stable if
  - The maximum Lyapunov exponent is negative
  - The CS pattern is asymptotically valid

- Coarser patterns may exist

- Multi-stability of coexisting attractors (with different basins of attractions)
An illustrative example

\[ \dot{x} = -y - z \]
\[ \dot{y} = x + ay \]
\[ \dot{z} = b + z(x - c) \]
An illustrative example

\[ \dot{x} = -y - z \]
\[ \dot{y} = x + ay \]
\[ \dot{z} = b + z(x - c) \]
An illustrative example
Larger networks (e.g., ER, N=200)
Beyond pairwise interactions

Modeling higher-order interactions

Simplicial complexes

Hypergraphs

General model of coupled oscillators with higher-order interactions

\[
\dot{x}_i = f(x_i) + \sigma_1 \sum_{j_1=1}^{N} a_{ij_1}^{(1)} g^{(1)}(x_i, x_{j_1}) \\
+ \sigma_2 \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} a_{ij_1j_2}^{(2)} g^{(2)}(x_i, x_{j_1}, x_{j_2}) + \ldots \\
+ \sigma_D \sum_{j_1=1}^{N} \ldots \sum_{j_D=1}^{N} a_{ij_1\ldots j_D}^{(D)} g^{(D)}(x_i, x_{j_1}, \ldots, x_{j_D}),
\]

- Information about the interactions is now encoded in tensors of order \( d \)

Global synchronization in simplicial complexes

- Existence of the synchronous solution:

\[ g^{(d)}(x, x, \ldots, x) \equiv 0 \ \forall d \]

- Conditions for stability (equations of the transverse modes):

\[
\begin{align*}
\dot{\eta}_1 &= JF \eta_1, \\
\dot{\eta}_i &= (JF - \sigma_1 \lambda_i JG^{(1)}) \eta_i - \sigma_2 \sum_{j=2}^{N} \tilde{L}_{ij}^{(2)} JG^{(2)} \eta_j - \ldots \\
&\quad - \sigma_D \sum_{j=2}^{N} \tilde{L}_{ij}^{(D)} JG^{(D)} \eta_j,
\end{align*}
\]

(Note that the modes are coupled as in temporal networks, multilayer networks, ...)
Generalized Laplacians
(there is always some)

\[
\mathcal{L}^{(d)}_{ij} = \begin{cases} 
0 & \text{for } i \neq j \text{ and } a_{ij}^{(1)} = 0 \\
-(d-1)! k^{(d)}_{ij} & \text{for } i \neq j \text{ and } a_{ij}^{(1)} = 1 \\
d! k^{(d)}_i & \text{for } i = j,
\end{cases}
\]

where

\[
k^{(d)}_{ij} = \frac{1}{(d-1)!} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_{d-1}=1}^{N} a^{(d)}_{i,i_1,i_2,...,i_{d-1}},
\]

is the number of d-simplices to which \((i, j)\) participates, and

\[
k^{(d)}_i = \frac{1}{d!} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_{d-1}=1}^{N} a^{(d)}_{i,i_1,i_2,...,i_d},
\]

is the generalized d-degree of node \(i\).
Examples of synchronization in simplicial complexes

- The region of synchronization depends on the structure of interactions and on the coupling functions (here for 2-body and 3-body interactions)

- The region is bounded (cases a, b, and c) or not (cases d and e)
Master Stability Function for simplicial complexes

• Diffusive coupling

\[ g^{(1)}(x_i, x_j) = h^{(1)}(x_j) - h^{(1)}(x_i), \]
\[ g^{(2)}(x_i, x_j, x_k) = h^{(2)}(x_j, x_k) - h^{(2)}(x_i, x_i), \]

• Natural coupling

\[ h^{(2)}(x, x) = h^{(1)}(x). \]

and so on for order d...

• Transverse modes now becomes decoupled

\[ \dot{\eta}_i = JF\eta_i, \]
\[ \dot{\eta}_i = (JF - \sigma_1\lambda_i JG^{(1)})\eta_i - \sigma_2 \sum_{j=2}^{N} \tilde{L}^{(2)}_{ij} JG^{(2)}\eta_j - \ldots \]
\[ - \sigma_D \sum_{j=2}^{N} \tilde{L}^{(D)}_{ij} JG^{(D)}\eta_j, \]

\[ \eta = \left[ Jf(x^s) - \alpha Jh^{(1)}(x^s) \right] \eta \]
Master Stability Function

\[ \dot{\eta} = \left[ Jf(x^s) - \alpha Jh^{(1)}(x^s) \right] \eta \]

• Complex networks

\[ \lambda_{\text{max}}(\alpha) < 0, \alpha = \sigma \lambda_h \]

\[ \forall \lambda_h \quad \text{eigenvalue of } L \]

• Simplicial complexes

\[ \lambda_{\text{max}}(\alpha) < 0, \alpha = \sigma_1 \lambda_h \]

\[ \forall \lambda_h \quad \text{eigenvalue of} \]

\[ M^{(D)} = \mathcal{L}^{(1)} + \frac{\sigma_2}{\sigma_1} \mathcal{L}^{(2)} + \cdots + \frac{\sigma_D}{\sigma_1} \mathcal{L}^{(D)} \]

![Graph showing \( \lambda_{\text{max}} \) vs. \( \alpha \) for different types of stability](image)
Master Stability Function for the Rössler oscillator
Cluster synchronization in simplicial complexes

\[ R_g L^{(d)} = L^{(d)} R_g \]

- The condition must hold at any order \( d \)

\[
\begin{align*}
R_g L^{(1)} &= L^{(1)} R_g \\
R_g L^{(2)} &= L^{(2)} R_g
\end{align*}
\]

\[
\begin{align*}
R_g L^{(1)} &= L^{(1)} R_g \\
R_g L^{(2)} &= L^{(2)} R_g
\end{align*}
\]
Hypergraphs vs. simplicial complexes

Links make the difference

Directionality in higher-order interactions

• Undirected hyperlink can be viewed as the collection of directed hyperlinks

\[
\begin{align*}
A^{(2)}_{i\pi(jk)} &= 1 & A^{(2)}_{i\pi(jk)} &= 1 & A^{(2)}_{j\pi(ik)} &= 0 & A^{(2)}_{j\pi(ik)} &= 0 & A^{(2)}_{k\pi(ij)} &= 1 & A^{(2)}_{k\pi(ij)} &= 0
\end{align*}
\]

• The analysis of synchronization stability can be readapted for the general case and for the natural coupling case as well

Directionality induced synchronization and desynchronization

Directionality induces desynchronization ($\sigma_1=0.001$, $\sigma_2=0.12$)

Directionality induces synchronization ($\sigma_1=0.01$, $\sigma_2=0.16$)
References and coauthors

