Are 2 derivatives enough to describe Nature at a fundamental level?

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Outline

• Why higher-order derivatives are usually not considered?

• Some known physical examples where higher-order derivatives are important

• Distinction between “fundamental” and “non-fundamental” higher-derivative equations

• Why fundamental interactions in Quantum Field Theory are described by Lagrangians with 1st and 2nd derivatives?

• What about (quantum) gravity?
Newton’s 2nd law

\[ m\ddot{x} = F(x, \dot{x}, t, ...) \]

Only 1° and 2° derivatives of position!
What about higher-order derivatives of position?

\[ m\ddot{x} = F(x, \dot{x}, t, \ldots) \]

Only 1° and 2° derivatives of position!

Beyond two derivatives

\[ \dddot{x} = \frac{d^3x(t)}{dt^3} \rightarrow \text{jerk} \]

\[ x^{(5)} = \frac{d^5x(t)}{dt^5} \rightarrow \text{crackle} \]

\[ x^{(4)} = \frac{d^4x(t)}{dt^4} \rightarrow \text{snap} \]

\[ x^{(6)} = \frac{d^6x(t)}{dt^6} \rightarrow \text{pop} \]

\[ \vdots \]
Beyond two derivatives

• In mechanical engineering, equations involving higher-order derivatives of position are very common!

• E.g., elevators, rollercoaster, space craft, ...

• Biomechanical effects due to high accelerations, jerk, snap,..., can be dangerous!

• Rollercoaster are built to have *acceleration* $\sim O(1 - 5g)$ and *jerk* $\sim$ constant

[Eager et al., Eur. J. Phys. 37, 065008]
Beyond two derivatives

\[ m\ddot{x} = F (x, \dot{x}, \ddot{x}, t, ...) \]

Why not?
Beyond two derivatives

\[ m\ddot{x} = F(x, \dot{x}, \ddot{x}, t, \ldots) \]

Why not?

Because differential equations involving higher-order time derivatives “typically” admit unstable solutions!
Beyond two derivatives

Example: 3rd order ‘oscillator’

\[ \tau \ddot{x} = \dot{x} + \omega^2 x, \quad \tau > 0 \]

• If \( \tau = 0 \) we recover the standard two-derivative harmonic oscillator

• If \( \tau \neq 0 \), we can use the ansatz \( x(t) = C e^{\alpha t} \), and working perturbatively in \( \tau \), we get

\[
x(t) = C_1 e^{i\omega t} e^{-\tau \omega^2 t/2} + C_2 e^{-i\omega t} e^{-\tau \omega^2 t/2} + C_3 e^{t/\tau}
\]

runaway solution!!!
Beyond two derivatives

Example: 4th order ‘oscillator’

\[ \tau^2 x^{(4)} = \ddot{x} + \omega^2 x, \quad \tau > 0 \]

• If \( \tau = 0 \) we recover the standard two-derivative harmonic oscillator

• If \( \tau \neq 0 \), we can use the ansatz \( x(t) = C \, e^{\alpha \, t} \), and working perturbatively in \( \tau \), we get

\[
  x(t) = C_1 e^{i(\omega - \tau^2 \omega^3/2)t} + C_2 e^{-i(\omega - \tau^2 \omega^3/2)t} \\
  + C_3 e^{-t/\tau} e^{-\tau \omega^2 t/2} + C_4 e^{+t/\tau} e^{+\tau \omega^2 t/2}
\]

runaway solution!!!
Beyond two derivatives: Hamiltonian

Example: \( L = L(x, \dot{x}) \), \( \frac{\partial^2 L}{\partial \dot{x}^2} \neq 0 \)

EOM: \[
\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0
\]

Conjugate variables: \( q = x, \quad p = \frac{\partial L}{\partial \dot{x}} \)

Hamiltonian: \( H = p \dot{x}(p) - L(q, \dot{x}(p)) \sim p^2 + \cdots \)

bounded from below!
Beyond two derivatives: Hamiltonian

Example: \( L = L(x, \dot{x}, \ddot{x}), \quad \frac{\partial^2 L}{\partial \ddot{x}^2} \neq 0 \)

EOM:
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\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{x}} = 0
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Beyond two derivatives: Hamiltonian

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\]

Conjugate variables:
\[
\begin{aligned}
q_1 &= x, & q_2 &= \dot{x}, \\
p_1 &= \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{x}}, & p_2 &= \frac{\partial L}{\partial \ddot{x}}
\end{aligned}
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Beyond two derivatives: Hamiltonian

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\end{align*}
\]

Hamiltonian:
\[
H = p_1 q_2 + p_2 \dot{x}(q_1, q_2, p_2) - L(q_1, q_2, \dot{x}(q_1, q_2, p_2)) \\
\sim p_1 q_2 + p_2^2 + \cdots
\]
Beyond two derivatives: Hamiltonian

Example: \( L = L(x, \dot{x}, \ddot{x}), \frac{\partial^2 L}{\partial \dot{x}^2} \neq 0 \)

EOM:
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Hamiltonian:
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H = p_1 q_2 + p_2 \ddot{x}(q_1, q_2, p_2) - L(q_1, q_2, \ddot{x}(q_1, q_2, p_2))
\sim p_1 q_2 + p_2^2 + \cdots
\]

If \( p_1 < 0 \) \implies \text{unbounded from below!} \implies \text{instability!}
If a non-degenerate Lagrangian, $L = L(x, \dot{x}, \ddot{x}, \ldots, x^{(n)})$, depends on the $n$-th derivative of a single configuration variable $x$, with $n > 1$, then the energy function in the corresponding Hamiltonian picture is unbounded from below.
Beyond two derivatives

BUT, higher-derivative equations often appear in physics!

We now show two examples of physical phenomena described in terms of equations containing higher-order time and/or space derivatives.

We use the electron as the test-object for our discussion:

• Electron at a classical level
• Electron at a quantum level
Beyond two derivatives

Classical theory of an electron

\[ m_0 \ddot{x}_\mu = e \dot{x}_\nu F^{self}_{\mu \nu} + F^{ext}_\mu, \quad (\dot{\cdot}) \equiv \frac{d}{ds} (\cdot) \]
Beyond two derivatives

Classical theory of an electron

\[ m_0 \ddot{x}_\mu = e \dot{x}_\nu F_{\mu \nu}^{\text{self}} + F_{\mu}^{\text{ext}}, \quad (\dot{\cdot}) \equiv \frac{d}{ds}(\cdot) \]

It can be recast as [Dirac force (1938)]

\[ m \ddot{x}_\mu = \frac{e^2}{6\pi c} \left( \dddot{x}_\mu + \frac{1}{c^2} \dot{x}_\mu \dddot{x}_\nu \dddot{x}^\nu \right) + F_{\mu}^{\text{ext}} \]

Non-relativistic limit [Abraham-Lorentz force (1909)]

\[ m \ddot{x} = \frac{e^2}{6\pi c} \dddot{x} + F^{\text{ext}}, \quad (\dot{\cdot}) \equiv \frac{d}{dt}(\cdot) \]
Beyond two derivatives

It describes the recoil force on an accelerating charged particle caused by the emitted electromagnetic radiation.
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Non-relativistic limit [Abraham-Lorentz force (1909)]

E.g. oscillator as external force:

\[ \ddot{x} = \tau \dot{x} - \omega^2 x, \quad \tau = \frac{e^2}{6\pi c m} \sim 10^{-23} \text{s} \]

\[ x(t) = C_1 e^{i\omega t} e^{-\lambda \omega^2 t/2} + C_2 e^{-i\omega t} e^{-\lambda \omega^2 t/2} + C_3 e^{t/\lambda} \]

physical solution  unphysical solution
Beyond two derivatives

It describes the recoil force on an accelerating charged particle caused by the emitted electromagnetic radiation.

Non-relativistic limit \[\text{[Abraham-Lorentz force (1909)]}\]

E.g. time-dependent external force:

\[
a(t) = \tau \ddot{a}(t) + F(t), \quad \tau = \frac{e^2}{6\pi c m} \sim 10^{-23} \text{s}
\]

\[
\Rightarrow \quad a(t) = -\frac{1}{m\tau} \int_{-\infty}^{t} dt' e^{\frac{t-t'}{\tau}} F(t') + a_0 e^{t/\tau}
\]

runaway solution!
Beyond two derivatives

\[ a(t) = -\frac{1}{m\tau} \int_{-\infty}^{t} dt' e^{\frac{t-t'}{\tau}} F(t') + a_0 e^{t/\tau} \]

If we choose the boundary condition:

\[ a_0 = \frac{1}{m\tau} \int_{-\infty}^{\infty} dt' e^{-\frac{t'}{\tau}} F(t') \]

We avoid the runaway solution:

\[ a(t) = \frac{1}{m\tau} \int_{t}^{\infty} dt' e^{\frac{t-t'}{\tau}} F(t') \]

BUT, causality is violated!
Beyond two derivatives

The runaway solution can be avoided by imposing a boundary condition.

In general, such a boundary condition violates causality at time scales

\[ \tau = \frac{e^2}{6\pi c m} \sim 10^{-23}\text{s} \]

Is this a problem?
Beyond two derivatives

The runaway solution can be avoided by imposing a boundary condition.

In general, such a boundary condition violates causality at time scales \( \tau = \frac{e^2}{6\pi c m} \sim 10^{-23}\, s \)

Is this a problem?

In my opinion, NO! Because this is just an effective classical description of the electron.

We should consider a quantum description according to which the state of an electron is described in terms of a dynamical wavefunction!
Beyond two derivatives

\[\tau = \frac{e^2}{6\pi \hbar} \frac{\hbar}{c} \sim \alpha \lambda c\]

\(\alpha = \) fine structure constant

\(\lambda = \) de Broglie wavelength
Beyond two derivatives

\[ \tau = \frac{e^2}{6 \pi c \hbar} \frac{\hbar}{c m} c \sim \alpha \lambda c \]

\( \alpha = \) fine structure constant
\( \lambda = \) de Broglie wavelength

Electron wavefunction

\[ \lambda \]
Beyond two derivatives

\[ \tau = \frac{e^2 \hbar}{6 \pi c \hbar c m} c \sim \alpha \lambda c \]

\( \alpha \) = fine structure constant
\( \lambda \) = de Broglie wavelength

Electron wavefunction

violation of causality
Beyond two derivatives

Higher-derivative equations usually appear in effective descriptions!

The same happens at a quantum level when working with effective-field-theories
Quantum description of an electron

Schrödinger equation:

\[ i\hbar \partial_t \psi(x, t) = \left( -\frac{\hbar^2}{2m} \nabla^2 + \cdots \right) \psi(x, t) = H \psi(x, t) \]
Quantum description of an electron

Schrödinger equation:

\[ i\hbar \partial_t \psi(x, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + \cdots \right] \psi(x, t) = H \psi(x, t) \]

\[ \downarrow \]

\[ i\partial_{ct} \psi(x, t) = \left[ -\frac{1}{2} \lambda^2 \nabla^2 + \cdots \right] \psi(x, t), \quad \lambda = \frac{\hbar}{mc} \]

wavelength
Schrödinger equation:

\[ i \partial_{ct} \psi(x, t) = \left[ -\frac{1}{2} \lambda^2 \nabla^2 + \cdots \right] \psi(x, t), \quad \lambda = \frac{\hbar}{mc} \]

Characteristic length scale \( L \) such that \( \nabla^2 \sim \frac{\partial^2}{\partial x^2} \sim \frac{1}{L^2} \):

\[ \lambda^2 \nabla^2 \sim \frac{\lambda^2}{L^2} < 1 \]
Beyond two derivatives

Schrödinger equation:

\[ i\partial_{ct}\psi(x, t) = \left[-\frac{1}{2}\lambda^2 \nabla^2 + \cdots \right]\psi(x, t), \quad \lambda = \frac{\hbar}{mc} \]

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\[ \lambda^2 \nabla^2 \sim \frac{\lambda^2}{L^2} < 1 \]

If \( \frac{\lambda^2}{L^2} \ll 1 \), we could consider a generalization of Schrödinger

\[ i\partial_{ct}\psi(x, t) = \left[-\frac{1}{2}\lambda^2 \nabla^2 + a\lambda^4 \nabla^4 + b\lambda^6 \nabla^6 + \cdots \right]\psi(x, t) = H\psi(x, t) \]
Beyond two derivatives

In coordinate space:

\[ H = \left[ -\frac{1}{2} \lambda^2 \nabla^2 + a \lambda^4 \nabla^4 + b \lambda^6 \nabla^6 + \cdots \right] \]

In momentum space:

\[ H = \left[ \frac{1}{2} \lambda^2 \vec{p}^2 + a \lambda^4 \vec{p}^4 - b \lambda^6 \vec{p}^6 + \cdots \right] \]

Since \( \lambda \sim \frac{1}{m} \), it means that we are doing a non-relativistic expansion for \( \vec{p}^2 \ll m^2 \).

The 2\textsuperscript{nd} term contributes to fine structure corrections, while the 3\textsuperscript{rd} term to hyperfine structure corrections.
Beyond two derivatives

The higher-derivative Schrödinger equation can be obtained as a non-relativistic expansion of the Dirac equation:

\[(i\hbar\gamma^\mu \partial_\mu + mc)\Psi(x) = 0\]
Beyond two derivatives

The higher-derivative Schrödinger equation can be obtained as a non-relativistic expansion of the Dirac equation:

\[(i\hbar\gamma^\mu \partial_\mu + mc)\Psi(x) = 0\]

The ‘relativistic’ Hamiltonian is given by:

\[H = c\sqrt{\hat{p}^2 + m^2 c^2} - mc^2\]

\[= \frac{1}{2m} \hat{p}^2 - \frac{1}{8c^2 m^3} \hat{p}^4 + \frac{1}{16c^4 m^5} \hat{p}^6 + \ldots\]

Dirac equation does NOT have higher derivatives!
Beyond two derivatives

Higher derivatives indeed appear in effective-field-theory descriptions!

Higher-derivative Schrödinger equation is an example of effective field theory!

However, the more fundamental equation – that is Dirac equation in this case – does NOT possess higher derivatives!

A similar story happens for other interactions in Nature, such as weak and strong interactions.
We should make a distinction between “fundamental” and “non-fundamental” equations.

Given a framework (e.g. Quantum Field theory) we define:

- **Fundamental equations** are those equations that describe the dynamics of fundamental degrees of freedom, and that can be derived from a Principle of Least Action.

- **Fundamental degrees of freedom** are those that are relevant at high energies and/or short distances (“elementary dofs”).

- Instead, **non-fundamental equations** are those that describe the dynamics of non-fundamental (low-energy) degrees of freedom.
"non-fundamental higher derivatives"

In the framework of QFT, electromagnetic, weak, and strong are fundamental interactions, and their dynamics is described by means of 1st and 2nd order equations!

Their Lagrangians are constrained by very few guiding principles:

Stability (unitarity), locality, strict renormalizability, symmetries
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In the framework of QFT, electromagnetic, weak, and strong are fundamental interactions, and their dynamics is described by means of $1^{\text{st}}$ and $2^{\text{nd}}$ order equations.

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Are 2 derivatives enough to describe Nature at a fundamental level?
Question

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YES!
Are 2 derivatives enough to describe Nature at a fundamental level?

YES!

BUT... is this also true for the gravitational interaction???
The gravitational interaction in General Relativity (GR) is described by a very simple action:

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \, R + S_m$$

The field equations are 2th order in derivatives:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}, \quad R \sim \partial^2 g$$

Very successful theory to describe gravitational physics in the “low-energy regime”
What about gravity?

BUT in the high-energy regime General Relativity breaks down!

If we describe the gravitational interaction within the framework of QFT, then the theory is non-renormalizable!!

Thus, at the quantum level General Relativity should be seen as a low-energy effective-field-theory description of gravity!
When gravity is turned on, higher-order derivatives are inevitable in a QFT approach!

E.g. one-loop integral (in pure gravity or with matter):

\[ \sim a R^2 + b R^{\mu\nu} R_{\mu\nu} \]

where the quadratic curvature terms contain 4th order derivatives

\[ R^2, R^{\mu\nu} R_{\mu\nu} \sim g \partial^4 g \]
What about gravity?

Can we find a strictly renormalizable QFT of gravity? (as we do for electromagnetic, weak and strong interactions?)
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Can we find a strictly renormalizable QFT of gravity? (as we do for electromagnetic, weak and strong interactions?)

The answer is YES! In four spacetime dimensions, there exists a ‘unique’ strictly renormalizable gravitational Lagrangian:

\[ S = \frac{1}{16\pi G} \int d^4 x \sqrt{-g} \left( R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} \right) \]

In addition to the spin-2 massless graviton, this theory contains a massive spin-0, and a massive spin-2.

Such a theory contains higher-order derivatives!!!
What about gravity?

\[ S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( \mathcal{R} + \alpha \mathcal{R}^2 + \beta \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} \right) \]

- If we assume that gravity can be described as a fundamental interaction in the framework of QFT,
- and require strict renormalizability,
- then, it follows that the fundamental equations must contain higher-order derivatives!
What about gravity?

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- If we assume that gravity can be described as a fundamental interaction in the framework of QFT,
- and require strict renormalizability,
- then, it follows that the fundamental equations must contain higher-order derivatives!

Given the above assumptions (within the framework of perturbative QFT), higher derivatives turn out to be fundamental when gravity is taken into account!
Are 2 derivatives enough to describe Nature at a fundamental level?
Question

Are 2 derivatives enough to describe Nature at a fundamental level?

NO!
Problem of quantum gravity

• The gravitational action quadratic in the curvature is strictly renormalizable:

\[ S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (\mathcal{R} + \alpha \mathcal{R}^2 + \beta \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}) \]

• BUT the theory is pathological because of Ostrogradsky instability!
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Spin-2 massive ghost \( \Rightarrow \) stability and unitarity are violated!
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• BUT the theory is pathological because of Ostrogradsky instability!

Spin-2 massive ghost \Rightarrow \text{stability and unitarity are violated!}

That’s, in my opinion, the main problem of Quantum Gravity!